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Cambridge International AS & A Level

MATHEMATICS (9709) P3

TOPIC WISE QUESTIONS + ANSWERS | COMPLETE SYLLABUS

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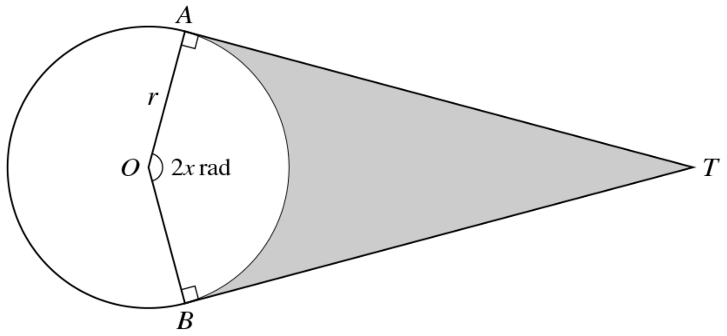
Chapter 6

Numerical solution of equations

214. 9709_s20_qp_31 Q: 6



The diagram shows a circle with centre O and radius r. The tangents to the circle at the points A and B meet at T, and angle AOB is $2x$ radians. The shaded region is bounded by the tangents AT and BT, and by the minor arc AB. The area of the shaded region is equal to the area of the circle.

(a) Show that x satisfies the equation $\tan x = \pi + x$. [3]

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(b) This equation has one root in the interval $0 < x < \frac{1}{2}\pi$. Verify by calculation that this root lies between 1 and 1.4. [2]

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(c) Use the iterative formula

$$x_{n+1} = \tan^{-1}(\pi + x_n)$$

to determine the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

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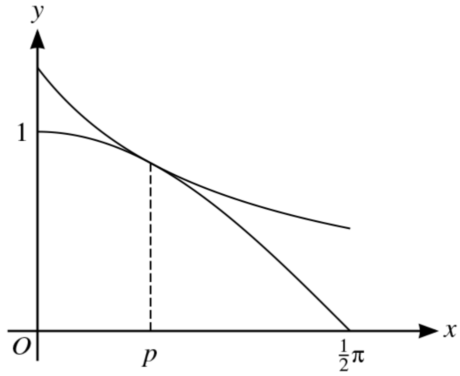
215. 9709_s20_qp_32 Q: 9



The diagram shows the curves $y = \cos x$ and $y = \frac{k}{1+x}$, where k is a constant, for $0 \leqslant x \leqslant \frac{1}{2}\pi$. The curves touch at the point where $x = p$.

(a) Show that p satisfies the equation $\tan p = \frac{1}{1+p}$. [5]

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(b) Use the iterative formula $p_{n+1} = \tan^{-1}\left(\frac{1}{1+p_n}\right)$ to determine the value of p correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

(c) Hence find the value of k correct to 2 decimal places. [2]

216. 9709_s20_qp_33 Q: 6

(a) By sketching a suitable pair of graphs, show that the equation $x^5 = 2 + x$ has exactly one real root. [2]

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(b) Show that if a sequence of values given by the iterative formula

$$x_{n+1} = \frac{4x_n^5 + 2}{5x_n^4 - 1}$$

converges, then it converges to the root of the equation in part **(a)**. [2]

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(c) Use the iterative formula with initial value $x_1 = 1.5$ to calculate the root correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

217. 9709_w20_qp_31 Q: 5

(a) By sketching a suitable pair of graphs, show that the equation $\operatorname{cosec} x = 1 + e^{-\frac{1}{2}x}$ has exactly two roots in the interval $0 < x < \pi$. [2]

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(b) The sequence of values given by the iterative formula

$$x_{n+1} = \pi - \sin^{-1}\left(\frac{1}{e^{-\frac{1}{2}x_n} + 1}\right),$$

with initial value $x_1 = 2$, converges to one of these roots.

Use the formula to determine this root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

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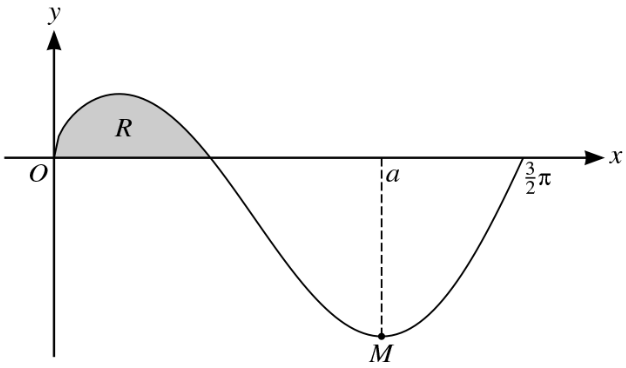
218. 9709_w20_qp_32 Q: 10



The diagram shows the curve $y = \sqrt{x}\cos x$, for $0 \leqslant x \leqslant \frac{3}{2}\pi$, and its minimum point M, where $x = a$. The shaded region between the curve and the x-axis is denoted by R.

(a) Show that a satisfies the equation $\tan a = \frac{1}{2a}$. [3]

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(b) The sequence of values given by the iterative formula $a_{n+1} = \pi + \tan^{-1}\left(\frac{1}{2a_n}\right)$, with initial value $x_1 = 3$, converges to a.

Use this formula to determine a correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

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(c) Find the volume of the solid obtained when the region R is rotated completely about the x-axis. Give your answer in terms of π. [6]

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219. 9709_m19_qp_32 Q: 2

The sequence of values given by the iterative formula

$$x_{n+1} = \frac{2x_n^6 + 12x_n}{3x_n^5 + 8},$$

with initial value $x_1 = 2$, converges to α.

(i) Use the formula to calculate α correct to 4 decimal places. Give the result of each iteration to 6 decimal places. [3]

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(ii) State an equation satisfied by α and hence find the exact value of α. [2]

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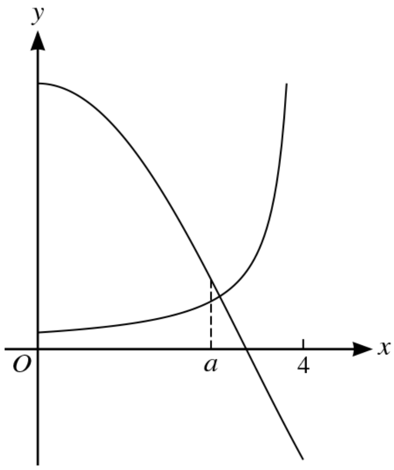
220. 9709_s19_qp_31 Q: 7



The diagram shows the curves $y = 4\cos\frac{1}{2}x$ and $y = \dfrac{1}{4-x}$, for $0 \leqslant x < 4$. When $x = a$, the tangents to the curves are perpendicular.

(i) Show that $a = 4 - \sqrt{(2\sin\frac{1}{2}a)}$. [4]

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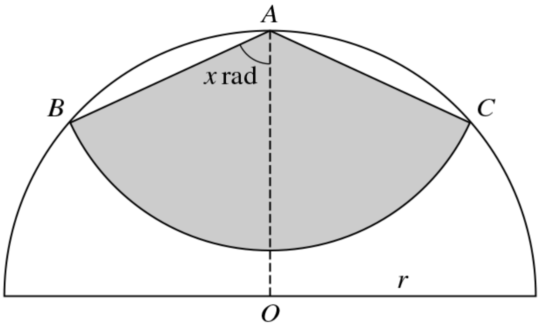
(ii) Verify by calculation that a lies between 2 and 3. [2]

(iii) Use an iterative formula based on the equation in part **(i)** to determine a correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

221. 9709_s19_qp_32 Q: 6



In the diagram, A is the mid-point of the semicircle with centre O and radius r. A circular arc with centre A meets the semicircle at B and C. The angle OAB is equal to x radians. The area of the shaded region bounded by AB, AC and the arc with centre A is equal to half the area of the semicircle.

(i) Use triangle OAB to show that $AB = 2r\cos x$. [1]

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(ii) Hence show that $x = \cos^{-1}\sqrt{\left(\frac{\pi}{16x}\right)}$. [2]

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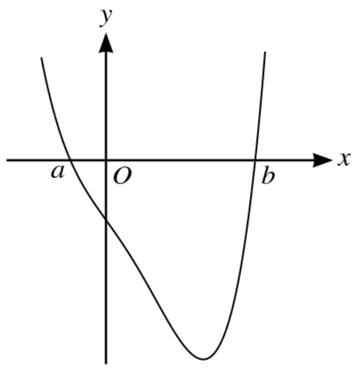
(iii) Verify by calculation that x lies between 1 and 1.5. [2]

(iv) Use an iterative formula based on the equation in part **(ii)** to determine x correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

222. 9709_s19_qp_33 Q: 6



The diagram shows the curve $y = x^4 - 2x^3 - 7x - 6$. The curve intersects the x-axis at the points $(a, 0)$ and $(b, 0)$, where $a < b$. It is given that b is an integer.

(i) Find the value of b. [1]

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(ii) Hence show that a satisfies the equation $a = -\frac{1}{3}(2 + a^2 + a^3)$. [4]

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(iii) Use an iterative formula based on the equation in part **(ii)** to determine a correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

223. 9709_w19_qp_31 Q: 5

The curve with equation $y = e^{-2x} \ln(x-1)$ has a stationary point when $x = p$.

(i) Show that p satisfies the equation $x = 1 + \exp\left(\frac{1}{2(x-1)}\right)$, where $\exp(x)$ denotes e^x. [3]

(ii) Verify by calculation that p lies between 2.2 and 2.6. [2]

(iii) Use an iterative formula based on the equation in part **(i)** to determine p correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

224. 9709_w19_qp_32 Q: 9

It is given that $\int_0^a x\cos\frac{1}{3}x\,dx = 3$, where the constant a is such that $0 < a < \frac{3}{2}\pi$.

(i) Show that a satisfies the equation

$$a = \frac{4 - 3\cos\frac{1}{3}a}{\sin\frac{1}{3}a}.$$ [5]

(ii) Verify by calculation that a lies between 2.5 and 3. [2]

(iii) Use an iterative formula based on the equation in part (i) to calculate a correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

225. 9709_w19_qp_33 Q: 5

(i) By sketching a suitable pair of graphs, show that the equation $\ln(x + 2) = 4e^{-x}$ has exactly one real root. [2]

(ii) Show by calculation that this root lies between $x = 1$ and $x = 1.5$. [2]

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(iii) Use the iterative formula $x_{n+1} = \ln\left(\frac{4}{\ln(x_n + 2)}\right)$ to determine the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

226. 9709_m18_qp_32 Q: 7

(i) By sketching suitable graphs, show that the equation $e^{2x} = 6 + e^{-x}$ has exactly one real root. [2]

(ii) Verify by calculation that this root lies between 0.5 and 1. [2]

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(iii) Show that if a sequence of values given by the iterative formula

$$x_{n+1} = \tfrac{1}{3}\ln\left(1 + 6e^{x_n}\right)$$

converges, then it converges to the root of the equation in part **(i)**. [2]

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(iv) Use this iterative formula to calculate the root correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

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227. 9709_s18_qp_31 Q: 8

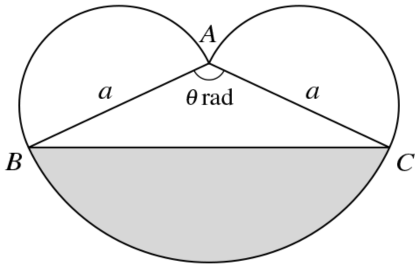
The positive constant a is such that $\int_0^a x\mathrm{e}^{-\frac{1}{2}x}\,\mathrm{d}x = 2$.

(i) Show that a satisfies the equation $a = 2\ln(a+2)$. [5]

(ii) Verify by calculation that a lies between 3 and 3.5. [2]

(iii) Use an iteration based on the equation in part (i) to determine a correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

228. 9709_s18_qp_32 Q: 6



The diagram shows a triangle ABC in which $AB = AC = a$ and angle $BAC = \theta$ radians. Semicircles are drawn outside the triangle with AB and AC as diameters. A circular arc with centre A joins B and C. The area of the shaded segment is equal to the sum of the areas of the semicircles.

(i) Show that $\theta = \frac{1}{2}\pi + \sin\theta$. [3]

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(ii) Verify by calculation that θ lies between 2.2 and 2.4. [2]

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(iii) Use an iterative formula based on the equation in part **(i)** to determine θ correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

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229. 9709_s18_qp_33 Q: 4

The curve with equation $y = \dfrac{\ln x}{3 + x}$ has a stationary point at $x = p$.

(i) Show that p satisfies the equation $\ln x = 1 + \dfrac{3}{x}$. [3]

(ii) By sketching suitable graphs, show that the equation in part **(i)** has only one root. [2]

(iii) It is given that the equation in part **(i)** can be written in the form $x = \dfrac{3+x}{\ln x}$. Use an iterative formula based on this rearrangement to determine the value of p correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

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230. 9709_w18_qp_31 Q: 3

(i) By sketching a suitable pair of graphs, show that the equation $x^3 = 3 - x$ has exactly one real root. [2]

(ii) Show that if a sequence of real values given by the iterative formula

$$x_{n+1} = \frac{2x_n^3 + 3}{3x_n^2 + 1}$$

converges, then it converges to the root of the equation in part **(i)**. [2]

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(iii) Use this iterative formula to determine the root correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

231. 9709_w18_qp_32 Q: 5

The equation of a curve is $y = x\ln(8 - x)$. The gradient of the curve is equal to 1 at only one point, when $x = a$.

(i) Show that a satisfies the equation $x = 8 - \dfrac{8}{\ln(8 - x)}$. [3]

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(ii) Verify by calculation that a lies between 2.9 and 3.1. [2]

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(iii) Use an iterative formula based on the equation in part **(i)** to determine a correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

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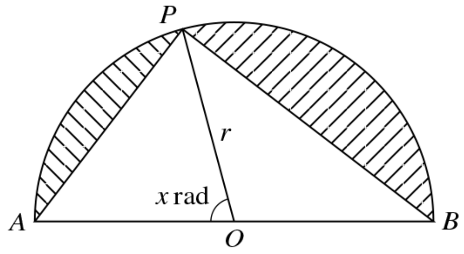
232. 9709_s17_qp_31 Q: 5



The diagram shows a semicircle with centre O, radius r and diameter AB. The point P on its circumference is such that the area of the minor segment on AP is equal to half the area of the minor segment on BP. The angle AOP is x radians.

(i) Show that x satisfies the equation $x = \frac{1}{3}(\pi + \sin x)$. [3]

(ii) Verify by calculation that x lies between 1 and 1.5. [2]

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(iii) Use an iterative formula based on the equation in part **(i)** to determine x correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

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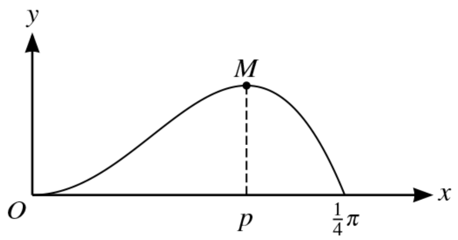
233. 9709_s17_qp_32 Q: 10



The diagram shows the curve $y = x^2 \cos 2x$ for $0 \leqslant x \leqslant \frac{1}{4}\pi$. The curve has a maximum point at M where $x = p$.

(i) Show that p satisfies the equation $p = \frac{1}{2}\tan^{-1}\left(\frac{1}{p}\right)$. [3]

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(ii) Use the iterative formula $p_{n+1} = \frac{1}{2}\tan^{-1}\left(\frac{1}{p_n}\right)$ to determine the value of p correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

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(iii) Find, showing all necessary working, the exact area of the region bounded by the curve and the x-axis. [5]

234. 9709_s17_qp_33 Q: 6

The equation $\cot x = 1 - x$ has one root in the interval $0 < x < \pi$, denoted by α.

(i) Show by calculation that α is greater than 2.5. [2]

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(ii) Show that, if a sequence of values in the interval $0 < x < \pi$ given by the iterative formula $x_{n+1} = \pi + \tan^{-1}\left(\frac{1}{1 - x_n}\right)$ converges, then it converges to α. [2]

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(iii) Use this iterative formula to determine α correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

235. 9709_w17_qp_31 Q: 3

The equation $x^3 = 3x + 7$ has one real root, denoted by α.

(i) Show by calculation that α lies between 2 and 3. [2]

Two iterative formulae, A and B, derived from this equation are as follows:

$$x_{n+1} = (3x_n + 7)^{\frac{1}{3}}, \qquad (A)$$

$$x_{n+1} = \frac{x_n^3 - 7}{3}. \qquad (B)$$

Each formula is used with initial value $x_1 = 2.5$.

(ii) Show that one of these formulae produces a sequence which fails to converge, and use the other formula to calculate α correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [4]

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236. 9709_w17_qp_32 Q: 9

It is given that $\int_1^a x^{\frac{1}{2}} \ln x \, dx = 2$, where $a > 1$.

(i) Show that $a^{\frac{3}{2}} = \dfrac{7 + 2a^{\frac{3}{2}}}{3 \ln a}$. [5]

(ii) Show by calculation that a lies between 2 and 4. [2]

(iii) Use the iterative formula

$$a_{n+1} = \left(\frac{7 + 2a_n^{\frac{3}{2}}}{3 \ln a_n} \right)^{\frac{2}{3}}$$

to determine a correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

237. 9709_m16_qp_32 Q: 3

The equation $x^5 - 3x^3 + x^2 - 4 = 0$ has one positive root.

(i) Verify by calculation that this root lies between 1 and 2. [2]

(ii) Show that the equation can be rearranged in the form

$$x = \sqrt[3]{\left(3x + \frac{4}{x^2} - 1\right)}.$$ [1]

(iii) Use an iterative formula based on this rearrangement to determine the positive root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

238. 9709_s16_qp_31 Q: 6

(i) By sketching a suitable pair of graphs, show that the equation

$$5e^{-x} = \sqrt{x}$$

has one root. [2]

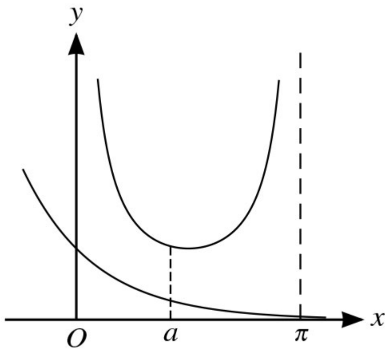
(ii) Show that, if a sequence of values given by the iterative formula

$$x_{n+1} = \frac{1}{2}\ln\left(\frac{25}{x_n}\right)$$

converges, then it converges to the root of the equation in part **(i)**. [2]

(iii) Use this iterative formula, with initial value $x_1 = 1$, to calculate the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

239. 9709_s16_qp_32 Q: 8



The diagram shows the curve $y = \operatorname{cosec} x$ for $0 < x < \pi$ and part of the curve $y = e^{-x}$. When $x = a$, the tangents to the curves are parallel.

(i) By differentiating $\dfrac{1}{\sin x}$, show that if $y = \operatorname{cosec} x$ then $\dfrac{dy}{dx} = -\operatorname{cosec} x \cot x$. [3]

(ii) By equating the gradients of the curves at $x = a$, show that

$$a = \tan^{-1}\left(\frac{e^a}{\sin a}\right).$$ [2]

(iii) Verify by calculation that a lies between 1 and 1.5. [2]

(iv) Use an iterative formula based on the equation in part **(ii)** to determine a correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

240. 9709_s16_qp_33 Q: 6

The curve with equation $y = x^2 \cos \frac{1}{2}x$ has a stationary point at $x = p$ in the interval $0 < x < \pi$.

(i) Show that p satisfies the equation $\tan \frac{1}{2}p = \dfrac{4}{p}$. [3]

(ii) Verify by calculation that p lies between 2 and 2.5. [2]

(iii) Use the iterative formula $p_{n+1} = 2\tan^{-1}\left(\dfrac{4}{p_n}\right)$ to determine the value of p correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

241. 9709_w16_qp_31 Q: 6

(i) By sketching a suitable pair of graphs, show that the equation

$$\operatorname{cosec} \tfrac{1}{2}x = \tfrac{1}{3}x + 1$$

has one root in the interval $0 < x \leqslant \pi$. [2]

(ii) Show by calculation that this root lies between 1.4 and 1.6. [2]

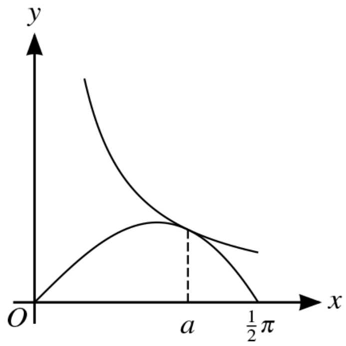
(iii) Show that, if a sequence of values in the interval $0 < x \leqslant \pi$ given by the iterative formula

$$x_{n+1} = 2\sin^{-1}\left(\frac{3}{x_n + 3}\right)$$

converges, then it converges to the root of the equation in part **(i)**. [2]

(iv) Use this iterative formula to calculate the root correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

242. 9709_w16_qp_33 Q: 9



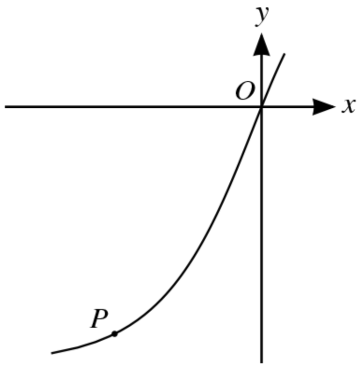
The diagram shows the curves $y = x\cos x$ and $y = \frac{k}{x}$, where k is a constant, for $0 < x \leqslant \frac{1}{2}\pi$. The curves touch at the point where $x = a$.

(i) Show that a satisfies the equation $\tan a = \frac{2}{a}$. [5]

(ii) Use the iterative formula $a_{n+1} = \tan^{-1}\left(\frac{2}{a_n}\right)$ to determine a correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

(iii) Hence find the value of k correct to 2 decimal places. [2]

243. 9709_s15_qp_31 Q: 10



The diagram shows part of the curve with parametric equations

$$x = 2\ln(t+2), \qquad y = t^3 + 2t + 3.$$

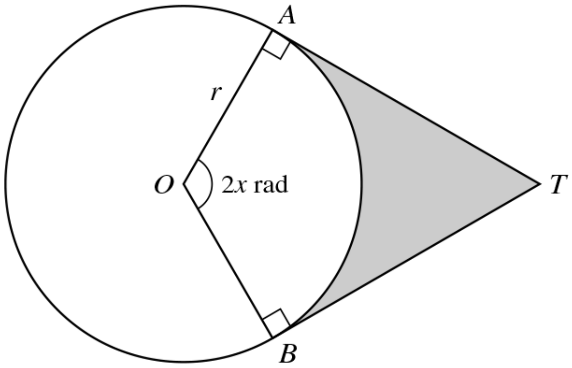
(i) Find the gradient of the curve at the origin. [5]

(ii) At the point P on the curve, the value of the parameter is p. It is given that the gradient of the curve at P is $\frac{1}{2}$.

(a) Show that $p = \dfrac{1}{3p^2+2} - 2$. [1]

(b) By first using an iterative formula based on the equation in part **(a)**, determine the coordinates of the point P. Give the result of each iteration to 5 decimal places and each coordinate of P correct to 2 decimal places. [4]

244. 9709_s15_qp_32 Q: 5



The diagram shows a circle with centre O and radius r. The tangents to the circle at the points A and B meet at T, and the angle AOB is $2x$ radians. The shaded region is bounded by the tangents AT and BT, and by the minor arc AB. The perimeter of the shaded region is equal to the circumference of the circle.

(i) Show that x satisfies the equation

$$\tan x = \pi - x.$$ [3]

(ii) This equation has one root in the interval $0 < x < \frac{1}{2}\pi$. Verify by calculation that this root lies between 1 and 1.3. [2]

(iii) Use the iterative formula

$$x_{n+1} = \tan^{-1}(\pi - x_n)$$

to determine the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

245. 9709_s15_qp_33 Q: 6

It is given that $\int_0^a x \cos x \, dx = 0.5$, where $0 < a < \frac{1}{2}\pi$.

(i) Show that a satisfies the equation $\sin a = \dfrac{1.5 - \cos a}{a}$. [4]

(ii) Verify by calculation that a is greater than 1. [2]

(iii) Use the iterative formula

$$a_{n+1} = \sin^{-1}\left(\frac{1.5 - \cos a_n}{a_n}\right)$$

to determine the value of a correct to 4 decimal places, giving the result of each iteration to 6 decimal places. [3]

246. 9709_w15_qp_31 Q: 4

The equation $x^3 - x^2 - 6 = 0$ has one real root, denoted by α.

(i) Find by calculation the pair of consecutive integers between which α lies. [2]

(ii) Show that, if a sequence of values given by the iterative formula

$$x_{n+1} = \sqrt{\left(x_n + \frac{6}{x_n}\right)}$$

converges, then it converges to α. [2]

(iii) Use this iterative formula to determine α correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

247. 9709_w15_qp_33 Q: 4

A curve has parametric equations

$$x = t^2 + 3t + 1, \qquad y = t^4 + 1.$$

The point P on the curve has parameter p. It is given that the gradient of the curve at P is 4.

(i) Show that $p = \sqrt[3]{(2p + 3)}$. [3]

(ii) Verify by calculation that the value of p lies between 1.8 and 2.0. [2]

(iii) Use an iterative formula based on the equation in part **(i)** to find the value of p correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]